TRAJECTORIES LAUNCHED NORMAL TO THE ECLIPTIC

by

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ABSTRACT

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A one-dimensional approximate description is given for the motion relative to earth of a vehicle launched normal to the ecliptic without excessive escape speed. A linearized perturbation treatment is provided for deviations from this one-dimensional motion, that are the result of the combined effects of the initial trajectory and the perturbations of moon and sun (the moon's orbit is regarded as circular in the ecliptic plane). The eccentricity of the earth's orbit is also disregarded here, and, on this basis, certain periodic motions which pass conveniently close to earth are found for this "restricted 4-body problem" of period 3 months. Author

1 INTRODUCTION

A vehicle launched from the Earth normal to the ecliptic enters a heliocentric orbit which returns to the vicinity of the Earth every 6 months. Except for appreciable "hyperbolic excess" speeds, however, escape from the Earth is not complete and the vehicle returns to Earth in less than 6 months. Such trips may be of interest for astronomical observation stations. This and other uses are discussed in Ref 1.

The analysis of the vehicle's trajectory will be based on the supposition that not only does the distance from Earth, measured in astronomical units, remain small, but that also the vehicle's direction from Earth remains approximately perpendicular to the ecliptic plane. More specifically, if a rotating coordinate system is introduced, centered at the Earth as indicated in Fig. 1, the coordinates x , y , z are assumed to be such that x , z $y << 1$. To the extent that $(x/y)^2$ and $(z/y)^2$ may be neglected, the y motion turns out to be uncoupled from the small x and z motions and may be studied separately.

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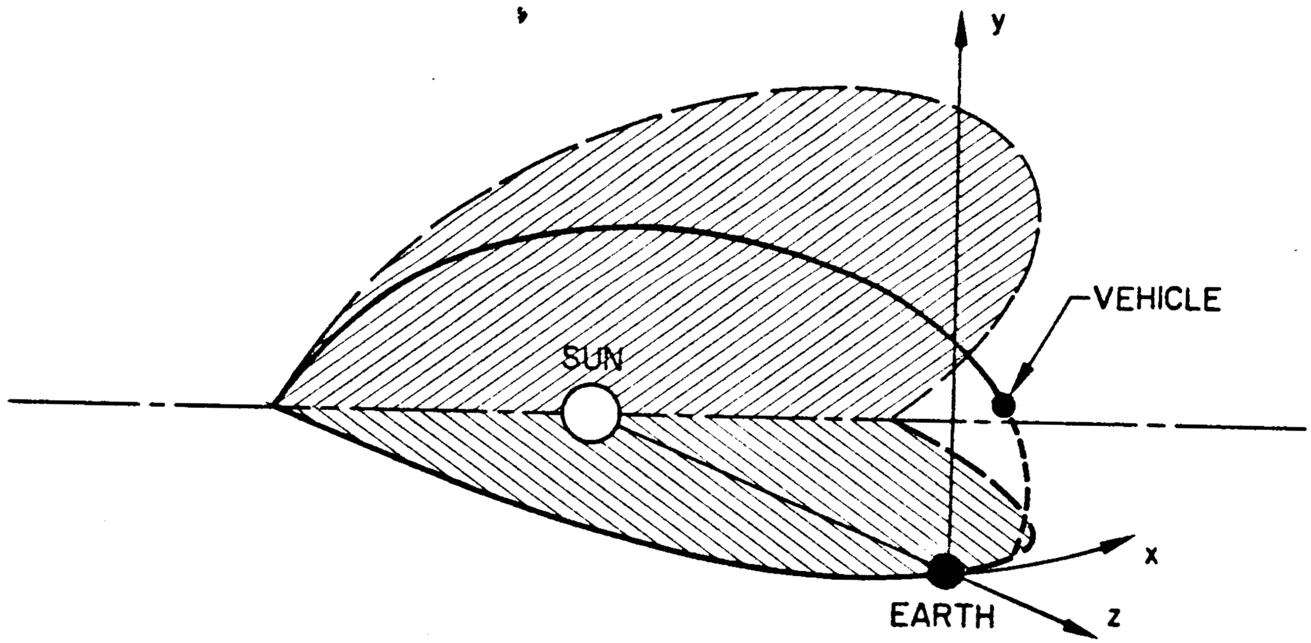


Fig. 1 Trajectory Geometry

After the equations of motion are worked out in general (in Section 2), the basic one-dimensional y motion for a zero-eccentricity Earth orbit is investigated in Section 3. As may be anticipated, this motion, which includes both the Sun's attraction and the Earth's attraction as "restoring" forces tending to reduce y to zero, depends on a single parameter — the one-dimensional total energy. It will be found (Fig. 2) that for total energies in the neighborhood of zero, in fact for orbit injection speeds differing from the theoretical escape speed by not more than 100 m/sec, the maximum distance from Earth varies from less than 1/2- to more than 7-1/2-million km, and the trip duration from less than 2 weeks to more than 5-1/2 months.

The linearized equation for the small "off-line" x and z motions are considered in Section 4, wherein y is now regarded as the 1-parameter function of time obtainable from Section 3. The acceleration $\ddot{z}$ contains, not surprisingly, a "driving term" proportional to y^2. The validity of the whole analysis depends, of course, on x/y and z/y remaining small when their "initial" values (i. e., shortly after launch) are small. The solution for the x and z motions will be obtained by variation of four small parameters, namely, the x- and z-direction cosines of the perigee of the geocentric conic and the x and z components of the geocentric angular momentum vector. The form of the equations of variation of parameters, Eqs. (23) and (24) below, suggests that the small parameters do indeed remain small at least for energies not too far in excess of zero (i. e., for values of the energy parameter η of Section 3 not too close to 1), i. e., for trips lasting not too close to 6 months. For trips lasting less than 5 months, this expectation has indeed been confirmed by the numerical evaluation. The linearized x and z motions will provide, in addition, an indication not only of the effect of launch errors but also of the effect of mid-course velocity vector changes.

The perturbations of the previous solutions by the eccentricity of the Earth's orbit is examined in Section 5, and those by the Moon in Section 6. In the latter case, the Moon is idealized as moving in a circle in the ecliptic plane, an idealization which may be expected to give a realistic first-order account, since the unperturbed motion is perpendicular to the ecliptic plane and the maximum angular departure of the Moon from this plane is less than 6 deg. The possibility of periodic orbits is also examined in Section 6.

2 THE EQUATIONS OF MOTION RELATIVE TO THE EARTH

Let $\vec{r}$ be the position vector of the vehicle relative to Earth, and $\vec{R}$ that of the Earth relative to the Sun. The acceleration of the vehicle relative to Earth is given by

$$\ddot{\vec{r}} = -\frac{GM_E\vec{r}}{r^3} - \left[\frac{GM_S(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3} - \frac{GM_S\vec{R}}{R^3}\right] \tag{1}$$

G being the universal gravitational constant and M_E, M_S the masses of Earth and Sun, respectively, and the square bracketed term being the "perturbative" acceleration due to the Sun.

Evaluating $\ddot{\vec{r}}$ in the rotating coordinate system indicated in Fig. 3-1, we have:

$$\ddot{\vec{r}} = \frac{\partial}{\partial t}\dot{\vec{r}} + \vec{\Omega}\times\dot{\vec{r}} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\vec{r} + \vec{\Omega}\times\vec{r}\right) + \vec{\Omega}\times\left(\frac{\partial}{\partial t}\vec{r} + \vec{\Omega}\times\vec{r}\right)$$

$$= \frac{\partial^2}{\partial t^2}\vec{r} + 2\vec{\Omega}\times\frac{\partial}{\partial t}\vec{r} + \frac{\partial\vec{\Omega}}{\partial t}\times\vec{r} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) \tag{2}$$

where $\frac{\partial}{\partial t}$ indicates time differentiation of the respective components, and $\vec{\Omega}$ the angular velocity of the rotating coordinate system. Introducing the components $(a_o x,\ a_o y,\ a_o z)$ of $\vec{r}$, where a_o is the Earth's mean distance from the Sun (=1 astronomical unit), and the components $(o,\ \dot{\theta},\ o)$ of $\vec{\Omega}$, θ being the Earth's heliocentric angular position along its orbit, measured from some fixed direction, as well as the components (o, o, R) of $\vec{R}$, we obtain from Eqs. (1) and (2)

$$\ddot{x} - \dot{\theta}^2 x + 2\dot{\theta}\dot{z} + \ddot{\theta} z = -\frac{GM_E x}{a_o^3\left(x^2+y^2+z^2\right)^{3/2}} - \frac{GM_s x}{\left[a_o^2\left(x^2+y^2\right)+\left(R+a_o z\right)^2\right]^{3/2}}$$

$$\ddot{y} = -\frac{GM_E y}{a_o^3\left(x^2+y^2+z^2\right)^{3/2}} - \frac{GM_s y}{\left[a_o^2\left(x^2+y^2\right)+\left(R+a_o z\right)^2\right]^{3/2}}$$

$$\ddot{z} - \dot{\theta}^2 z - 2\dot{\theta}\dot{x} - \ddot{\theta} x = -\frac{GM_E z}{a_o^3\left(x^2+y^2+z^2\right)^{3/2}} - \frac{GM_s (R+a_o z)}{a_o\left[a_o^2\left(x^2+y^2\right)+\left(R+a_o z\right)^2\right]^{3/2}} + \frac{GM_s}{a_o R^2} \qquad (3)$$

We now introduce as an independent variable, in place of time, the mean anomaly ϕ of the Earth's motion, so that $R \simeq a_o(1-\epsilon_E \cos\phi)$, ϵ_E being the eccentricity of the Earth's orbit, and

$$\frac{d}{d\phi} = \left(\frac{a_o^3}{GM_S}\right)^{1/2} \frac{d}{dt}$$

Expanding in ascending powers of y, $\frac{x}{y}$, $\frac{z}{y}$, the second Eq. (3) becomes:

$$\frac{d^2 y}{d\phi^2} + y(1+3\epsilon_E \cos\phi) + \frac{\rho}{y^2} = 0, \qquad (4)$$

where

$$\rho = \frac{M_E}{M_S} \; (<< 1), \qquad (5)$$

and where, in accordance with the smallness of ϵ_E and with our underlying assumption: $x, z << y << 1$, we have neglected ϵ_E^2, yz, y^3, $\frac{\rho x^2}{y^2}$, etc. The first and last Eq. (3), moreover, linearized with respect to x and z, become:

$$\left.\begin{aligned} \frac{d^2x}{d\varphi^2} + 2\frac{dz}{d\varphi} + \frac{\rho x}{y^3} &= 0 \\ \frac{d^2z}{d\varphi^2} - 3z - 2\frac{dx}{d\varphi} + \frac{\rho z}{y^3} &= \frac{3}{2}y^2 \end{aligned}\right\} \qquad (6)$$

Here we have neglected y^4 and products of ϵ_E or y^2 with the small quantities x and z, which products include the $\ddot{\theta}$ terms in Eq. (3. 3).

3 BASIC ONE-DIMENSIONAL MOTION

Neglecting ϵ_E in Eq. (4) (it will be reintroduced in Section 5), we have:

$$\frac{d^2y}{d\varphi^2} + y + \frac{\rho}{y^2} = 0 \qquad (7)$$

We immediately obtain an "energy" integral:

$$\frac{1}{2}\left(\frac{dy}{d\varphi}\right)^2 + \frac{1}{2}y^2 - \frac{\rho}{y} = E \qquad (8)$$

which is a constant.

A further quadrature yields the time (through φ) as follows:

$$\varphi - \varphi_o = \int_o^y \frac{\sqrt{y}\, dy}{\sqrt{2\rho + 2Ey - y^3}} \qquad (9)$$

where the lower limit is taken as zero, rather than some initial $y_o << \rho^{1/3}$ (at which distance the perturbing force due to the Sun is much smaller than the Earth's attraction), so that the contribution to ϕ between o and y_o is unimportant.

A convenient description of the motion, involving essentially one parameter, is obtained if we introduce y_1, the maximum y reached, given by:

$$2\rho + 2Ey_1 - y_1^3 = 0 ,$$

and a nondimensional energy parameter η given by:

$$\eta = \frac{2E}{y_1^2} = 1 - \frac{2\rho}{y_1^3} , \tag{10}$$

so that

$$y_1 = \left(\frac{2\rho}{1-\eta}\right)^{1/3} \tag{11}$$

As the parameter η varies from $-\infty$ to $+1$, the maximum distance y_1 (in astronomical units) varies from 0 to ∞. Our solution, of course, is valid only over the range of values of η for which $y_1 << 1$ (e.g., $y_1 < 0.1$).

The Earth's position during the outward motion of the vehicle is given by:

$$\phi - \phi_o = \int_o^{(y/y_1)} \frac{\sqrt{u}\, du}{\sqrt{(1-u)(1-\eta+u+u^2)}} , \tag{12}$$

and the total angular travel of the Earth, and hence the total time, by, say,

$$\phi_2 - \phi_o = 2(\phi_1 - \phi_o) = 2\int_o^1 \frac{\sqrt{u}\ du}{\sqrt{(1-u)(1-\eta+u+u^2)}} = f(\eta) \qquad (13)$$

In particular, $f\left(\frac{3}{4}\right) = \frac{2\pi}{\sqrt{3}}\left(\sqrt{3}-1\right)$. Other values of $f(\eta)$ may be obtained from tables of elliptic functions. Thus if $\eta < \frac{3}{4}$, $f(\eta)$ may be evaluated according to 3(d) on p. 48 of Gröbner and Hofreiter's Integral Tafeln, Vol 2, while if $\eta > \frac{3}{4}$, $f(\eta)$ may be evaluated by 3(a) on p. 67.

The parameter η, in turn, may be related to the speed v_o at a specified small radial distance r_o from Earth, at which distance the potential energy term $\frac{1}{2}y^2$ due to the Sun's perturbative force may be assumed to be negligible by comparison with the potential energy term $-\rho/y$ due to the Earth's attraction, so that:

$$\frac{1}{2}v_o^2 - \frac{GM_E}{r_o} = \frac{GM_s}{a_o}\left[\frac{1}{2}\left(\frac{dy}{d\phi}\right)^2 - \frac{\rho}{y}\right] \cong \frac{GM_s}{a_o}E$$

(at $y << \rho^{1/3}$)

$$= \frac{GM_s}{a_o}\frac{\eta y_1^2}{2} = \frac{GM_E}{r_o}\cdot\frac{1}{\rho}\frac{r_o}{a_o}\frac{\eta}{2}\left(\frac{2\rho}{1-\eta}\right)^{2/3}$$

and hence, assuming that v_o is close to $\sqrt{\frac{2GM_E}{r_o}}$:

$$v_o \cong \sqrt{\frac{2GM_E}{r_o}}\left\{1 + \frac{\eta}{\left[2(1-\eta)^2\right]^{1/3}}\ \frac{1}{\rho^{1/3}}\left(\frac{r_o}{a_o}\right)\right\} \qquad (14)$$

Equations (11), (13), and (14) may be used to plot the relations between maximum distance, "initial" speed, and flight duration. Thus, in Fig. 2 the ordinate is $f(\eta)$, the Earth's angular travel, the lower abscissa is $(1-\eta)^{-1/3}$ (or, rather, its logarithm), and the upper abscissa is $\eta/\left[2(1-\eta)^2\right]^{1/3} = g(\eta)$, say, so that by suitable rescaling we can read the maximum distance $a_o y_1$, in millions of km, $\Delta v_o = v_o - \sqrt{\frac{2GM_E}{r_o}}$ in m/sec, with r_o chosen as 150 km in excess of the Earth's equatorial radius, versus the flight duration in days.

In Table 1 of Ref. 1, it is shown how to rescale Fig. 2 to make it apply to trips launched from other planets perpendicular to their heliocentric orbital planes or from the moon perpendicular to its orbital plane.

The sensitivity of the flight duration and the maximum distance reached to initial speed is apparent from Fig. 2. To evaluate the sensitivity of the return time to mid-course changes, small or otherwise, in the velocity in the y direction, it is necessary to evaluate the intermediate times by means of Eq. (12). For both the outward and return journeys, these times are evaluated by numerical integration, after chosing, in place of u, a new independent variable $s = \pm\sqrt{1-u}$, according as $\dot{y} \gtrless 0$. (See Section 4 below.) The results are shown in Fig. 3.

4 OFF-LINE MOTIONS

Before attacking the Eq. (6) of off-line motion, let us examine the significance of the assumption x, $z << y$ in the neighborhood of the Earth.

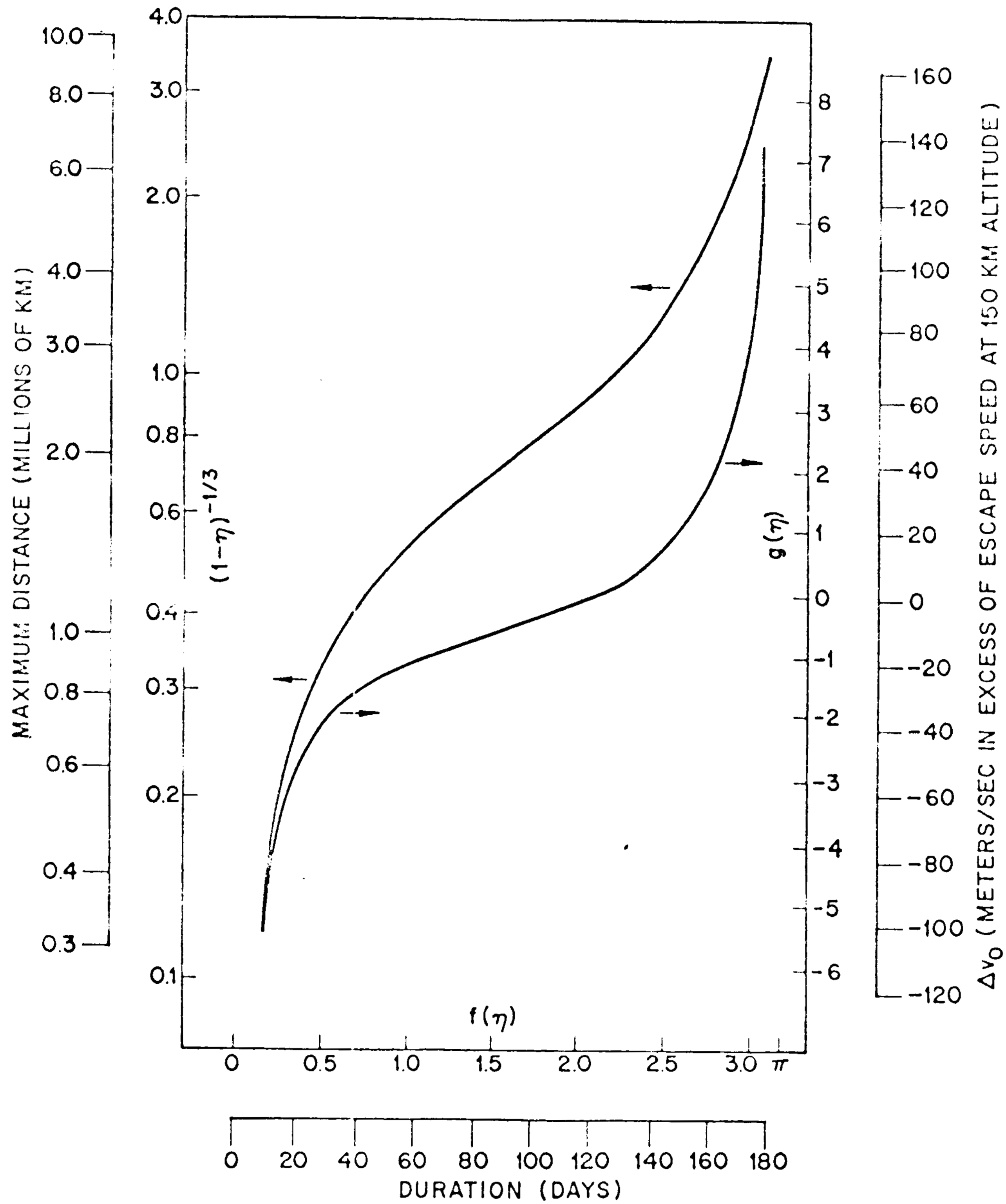


Fig. 2 Initial Speed and Maximum Distance Versus Duration

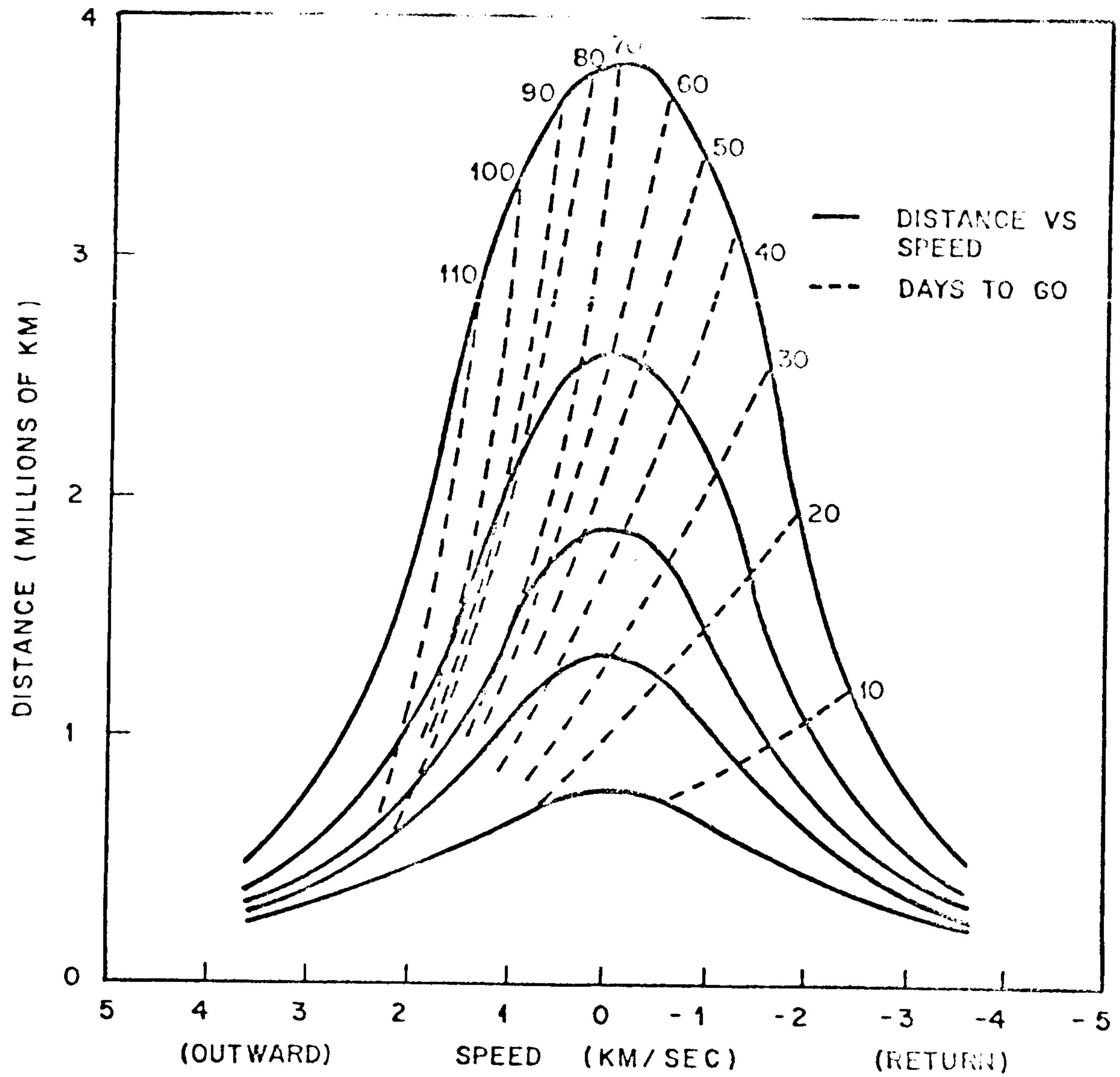
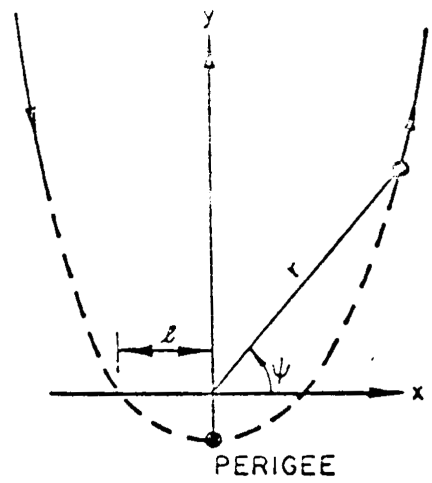


Fig. 3 Motion Histories

Consider first a conic (Fig. 4) in the xy-plane, with semi-latus rectum ℓ , perigee in the (-y)-direction, and eccentricity ϵ close to 1 (it may be more or less than 1). Its equation is:



$$a_o\sqrt{x^2+y^2} = r = \frac{\ell}{1-\epsilon\sin\psi} = \frac{\ell}{1-\epsilon\dfrac{y}{\sqrt{x^2+y^2}}}$$

so that $\sqrt{x^2+y^2} = \epsilon y + \ell/a_o$, and hence:

Fig. 4 Nearly Parabolic Conic Near the Earth

$$x^2 = \left(\frac{\ell}{a_o}\right)^2 + 2\epsilon\left(\frac{\ell}{a_o}\right)y + \left(\epsilon^2-1\right)y^2 \cong \left(\frac{\ell}{a_o}\right)^2 + 2\left(\frac{\ell}{a_o}\right)y + (\epsilon^2-1)y^2.$$

That part of the conic for which $x << y$, and has a $y >> \left(\frac{\ell}{a_o}\right)$ (the full line in Fig. 4), thus satisfies:

$$x \cong \pm y\sqrt{\epsilon^2 - 1 + \frac{2\ell}{a_o y}}$$

Furthermore, the velocity along the conic is given by

$$a_o^2\left(\dot{x}^2+\dot{y}^2\right) = v^2 = \frac{2GM_E}{r} + \frac{GM_E\left(\epsilon^2-1\right)}{\ell} = \frac{GM_E}{\ell}\left(\epsilon^2 - 1 + \frac{2\ell}{a_o\sqrt{x^2+y^2}}\right)$$

which becomes, if we neglect x^2 and $\dot{x}^2$

$$\dot{y} = \pm\frac{1}{a_o}\sqrt{\frac{GM_E}{\ell}}\sqrt{\epsilon^2 - 1 + \frac{2\ell}{a_o y}}$$

The part of the conic for which $x << y$ thus satisfies:

$$x \cong a_o \left(\frac{\ell}{GM_E}\right)^{1/2} y\,\dot{y} = \left(\frac{1}{\rho}\frac{\ell}{a_o}\right)^{1/2} y \frac{dy}{d\phi}$$

Note that the coefficient here of $y \frac{dy}{d\phi}$ is, apart from a fixed scale factor, just the angular momentum $\sqrt{GM_E \ell}$ of the geocentric motion.

The generalization to any nearly parabolic conic whose axis nearly coincides with the y-axis is clear; if $x, \; z << y$

$$\left.\begin{aligned} x &= \left(\frac{1}{\rho}\frac{\ell}{a_o}\right)^{1/2} \cos\alpha \; y \frac{dy}{d\phi} + Ay, \\ z &= \left(\frac{1}{\rho}\frac{\ell}{a_o}\right)^{1/2} \sin\alpha \; y \frac{dy}{d\phi} + By, \end{aligned}\right\} \qquad (15)$$

where α is the angle which the angular momentum vector makes with the z-axis (measured toward the negative x-axis), and A, B are the (small) direction cosines of the conic's axis in the x-, z directions respectively, i. e., $(-A)$ and $(-B)$ are the x- and z- direction cosines of the conic's perigee direction.

The foregoing discussion of the motion near Earth suggests the following "variation of parameters" description of the entire motion:

$$\left.\begin{aligned} x &= Cy\frac{dy}{d\phi} + Ay \quad , \quad \frac{dx}{d\phi} = C\left[\left(\frac{dy}{d\phi}\right)^2 + y\frac{d^2y}{d\phi^2}\right] + A\frac{dy}{d\phi} \\ z &= Dy\frac{dy}{d\phi} + By \quad , \quad \frac{dz}{d\phi} = D\left[\left(\frac{dy}{d\phi}\right)^2 + y\frac{d^2y}{d\phi^2}\right] + B\frac{dy}{d\phi}, \end{aligned}\right\} \qquad (16)$$

where A, B, C, D are now variables which vary, however, very little in passing from launch at perigee, for example, to a near region, as in Fig. 4, where x, z << y, and where $\frac{d^2y}{d\varphi^2}$ is now given by Eq. (7), thus including the Sun's perturbative force as well as the Earth's attraction. Note that, accordingly, the rescaled angular momentum components, $x\frac{dy}{d\varphi} - y\frac{dx}{d\varphi}$ and $z\frac{dy}{d\varphi} - y\frac{dz}{d\varphi}$, are respectively $C(\rho + y^3)$ and $D(\rho + y^3)$, which can be regarded as fixed multiples of C and D only in the near-Earth region $y << \rho^{1/3}$.

Differentiation of the left Eq. (16) in combination with the right equations yields:

$$\frac{dA}{d\varphi} = -\frac{dy}{d\varphi}\frac{dC}{d\varphi} \quad \text{and} \quad \frac{dB}{d\varphi} = -\frac{dy}{d\varphi}\frac{dD}{d\varphi} \quad . \tag{17}$$

Substitution of Eq. (16) into Eq. (6), together with Eqs. (7), (8), and (17), yields:

$$\begin{aligned} \frac{dC}{d\varphi} &= \frac{y}{\rho + y^3}\left\{2\left(2E + \frac{\rho}{y} - 2y^2\right)D + 2\frac{dy}{d\varphi}B - 4y\frac{dy}{d\varphi}C - yA\right\} \\ \frac{dD}{d\varphi} &= \frac{y}{\rho + y^3}\left\{-7y\frac{dy}{d\varphi}D - 4yB - 2\left(2E + \frac{\rho}{y} - 2y^2\right)C - 2\frac{dy}{d\varphi}A - \frac{3}{2}y^2\right\} \end{aligned} \tag{18}$$

A convenient renormalization of these equations is obtained by introducing, as at the end of Section 3, a new independent variable:

$$s = \mp\sqrt{1 - \frac{y}{y_1}} = \mp\sqrt{1 - u} \quad , \text{ according as } \frac{dy}{d\varphi} \gtrless 0, \tag{19}$$

so that s increases monotonically from -1 to $+1$ between the time that the vehicle leaves the vicinity of the Earth and the time of return to Earth. From Eq. (12) we obtain

$$\frac{ds}{d\varphi} = \frac{1}{2\sqrt{u}}\sqrt{1-\eta+u+u^2} \tag{20}$$

At the same time we choose the following variable small parameters:

$$\left.\begin{aligned}
\lambda_1 &= \left(\frac{1-\eta}{2}\right)^{1/3} y_1 C = -\frac{\rho^{-1/6}}{1+\frac{2}{1-\eta}\left(\frac{y}{y_1}\right)^3}\left(\frac{\ell}{a_o}\right)^{1/2}\cos\left(\Omega-\lambda_s\right)\\
\lambda_2 &= A = \left(\frac{\pi}{2}-i\right)\sin\left(\Omega-\lambda_s\right)-\left(\frac{3\pi}{2}-\omega\right)\cos\left(\Omega-\lambda_s\right)\\
\lambda_3 &= \left(\frac{1-\eta}{2}\right)^{1/3} y_1 D = -\frac{\rho^{-1/6}}{1+\frac{2}{1-\eta}\left(\frac{y}{y_1}\right)^3}\left(\frac{\ell}{a_o}\right)^{1/2}\sin\left(\Omega-\lambda_s\right)\\
\lambda_4 &= B = -\left(\frac{\pi}{2}-i\right)\cos\left(\Omega-\lambda_s\right)-\left(\frac{3\pi}{2}-\omega\right)\sin\left(\Omega-\lambda_s\right)
\end{aligned}\right\} \tag{21}$$

Here i, Ω, ω, and ℓ are the inclination, nodal longitude, argument of perigee and semi-latus rectum of the instantaneous earth-centered conic (referred to the ecliptic rather than equator), so that ℓ/a_o, $\pi/2 - i$ and $3\pi/2 - \omega$ are small quantities.

Substitution of Eq. (21) into Eqs. (17) and (18) together with Eqs. (8), (10), (11), (19), and (20) yields

$$\left.\begin{aligned}
\frac{d\lambda_1}{ds} &= \sqrt{u}\;\Phi_1\\
\frac{d\lambda_2}{ds} &= \left(\frac{2}{1-\eta}\right)^{1/3} s\sqrt{1-\eta+u+u^2}\;\Phi_1\\
\frac{d\lambda_3}{ds} &= \sqrt{u}\;\Phi_2\\
\frac{d\lambda_4}{ds} &= \left(\frac{2}{1-\eta}\right)^{1/3} s\sqrt{1-\eta+u+u^2}\;\Phi_2
\end{aligned}\right\} \tag{22}$$

where

$$\Phi_1 = \frac{4s\sqrt{u}}{1-\eta+2u^3}\left[4u\lambda_1 - 2\left(\frac{1-\eta}{2}\right)^{1/3}\lambda_4\right] + \frac{4}{(1-\eta+2u^3)\sqrt{1-\eta+u+u^2}}$$

$$\left[\left(1-\eta+2\eta u-4u^3\right)\lambda_3 - \left(\frac{1-\eta}{2}\right)^{1/3}u^2\lambda_2\right]$$

$$\Phi_2 = \frac{4s\sqrt{u}}{1-\eta+2u^3}\left[7u\lambda_3 + 2\left(\frac{1-\eta}{2}\right)^{1/3}\lambda_2\right] - \frac{4}{(1-\eta+2u^3)\sqrt{1-\eta+u+u^2}}$$

$$\left[\left(1-\eta+2\eta u-4u^3\right)\lambda_1 + 4\left(\frac{1-\eta}{2}\right)^{1/3}u^2\lambda_4\right]$$

$$- \frac{6\rho^{1/3}u^3}{(1-\eta+2u^3)\sqrt{1-\eta+u+u^2}} \qquad (23)$$

The solution of Eqs. (22) and (23), which will be obtained on a digital computer, clearly has the form:

$$\begin{pmatrix} \lambda_1(s) \\ \lambda_2(s) \\ \lambda_3(s) \\ \lambda_4(s) \end{pmatrix} = T(\eta, s)\begin{pmatrix} \lambda_1(-1) \\ \lambda_2(-1) \\ \lambda_3(-1) \\ \lambda_4(-1) \end{pmatrix} + \rho^{1/3}\, S(\eta, s)\,, \qquad (24)$$

where $T(\eta, s)$ is a 4×4 transition matrix, representing the propagation of initial "errors," with initial value:

$$T(\eta, -1) = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix},$$

The condition for a direct collision with the Earth, of course, is just: $\lambda_1(1) = \lambda_3(1) = 0$.

The inverse of Eq. (25) combined with Eq. (24) yields the sensitivities of the instantaneous x, $\dot{x}$, z, $\dot{z}$ to changes in the initial small parameters $\lambda_i(-1)$

$$\begin{pmatrix} \delta x \\ \delta \dot{x} \\ \delta z \\ \delta \dot{z} \end{pmatrix} = \begin{pmatrix} N^{-1} & O \\ O & N^{-1} \end{pmatrix} T(\eta, s) \begin{pmatrix} \delta\lambda_1(-1) \\ \delta\lambda_2(-1) \\ \delta\lambda_3(-1) \\ \delta\lambda_4(-1) \end{pmatrix}, \qquad (27)$$

which, together with the sensitivities of y, $\dot{y}$ to the initial time t_o and energy parameter η, form the basis for a differential-correction of the 6 parameters t_o, η, $\lambda_1(-1)$, $\lambda_2(-1)$, $\lambda_3(-1)$, $\lambda_4(-1)$, to fit observations such as measurements of angles, ranges or range-rates.

5 EFFECT OF THE ECCENTRICITY ϵ_E OF THE EARTH'S ORBIT

Returning to Eq. (4) for the one-dimensional y-motion, with ϵ_E now included as a small parameter, let φ_1 be the eccentric anomaly of the Earth's position at the moment when the vehicle reaches its maximum distance y_1. Then, neglecting ϵ_E^2, we obtain

$$\left(\frac{dy}{d\varphi}\right)^2 = \frac{2\rho}{y} - \frac{2\rho}{y_1} - y^2 + y_1^2 - 6\epsilon_E \int_{y_1}^{y} y \cos\varphi \, dy , \qquad (28)$$

where the last term of ϕ is computed as a function of y from the "unperturbed" motion corresponding to $\epsilon_E = 0$; i.e., according to Eqs. (19) and (20)

$$\phi - \phi_1 = \int_0^s \frac{2\sqrt{u}\, ds}{\sqrt{1-\eta+u+u^2}} \tag{29}$$

where $u = \frac{y}{y_1} = 1 - s^2$.

The correction in the instantaneous $\left|\frac{dy}{d\phi}\right|$ is thus given by

$$\frac{\delta\left(\left|\frac{dy}{d\phi}\right|\right)}{\left|\frac{dy}{d\phi}\right|} = \nu, \text{ say}, = \delta\epsilon_E \cos\varphi_1 \frac{u\int_0^s us\cos(\phi-\varphi_1)\, ds}{s^2(1-\eta+u+u^2)} - \delta\epsilon_E \sin\phi_1 \frac{u\int_0^s us\sin(\phi-\phi_1)\, ds}{s^2(1-\eta+u+u^2)} \tag{30}$$

in which $\phi - \phi_1$ is given by Eq. (29).

The velocity correction given by Eq. (30) integrates to a time correction given by

$$\delta\phi = -2\int_0^s \frac{\nu\sqrt{u}}{\sqrt{1-\eta+u+u^2}}\, ds \tag{31}$$

to be added to the φ obtainable from Eq. (29). The velocity and time corrections ν and $\delta\phi$ will be obtained by numerical integration along with φ, $T(\eta, s)$ and $S(\eta, s)$.

6 EFFECT OF THE MOON

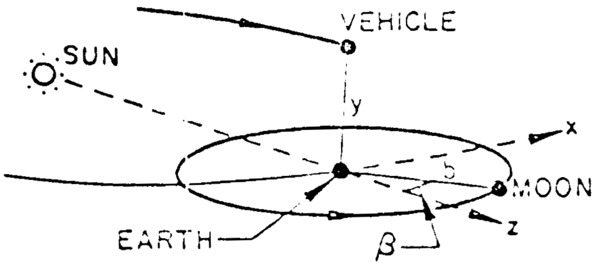


Fig. 5 The Presence of the Moon

If, as shown in Fig. 5, we idealize the Moon's orbit as a circle of radius b astronomical units in the x-z plane, and if we denote the Moon's "phase," (i.e., its angular position east of the midnight position) by β, and its mass by M_M, the components of the perturbative force of the moon on the vehicle in its unperturbed one-dimensional motion, $x = z = 0$, are

$$
\left.
\begin{aligned}
F_x^{(M)} &= \frac{GM_M b \sin\beta}{a_o^2\left(b^2+y^2\right)^{3/2}} - \frac{GM_M \sin\beta}{a_o^2 b^2} \, , \\
F_y^{(M)} &= -\frac{GM_M y}{a_o^2\left(b^2+y^2\right)^{3/2}} \, , \\
F_z^{(M)} &= \frac{GM_M b \cos\beta}{a_o^2\left(b^2+y^2\right)^{3/2}} - \frac{GM_M \cos\beta}{a_o^2 b^2}
\end{aligned}
\right\} \qquad (32)
$$

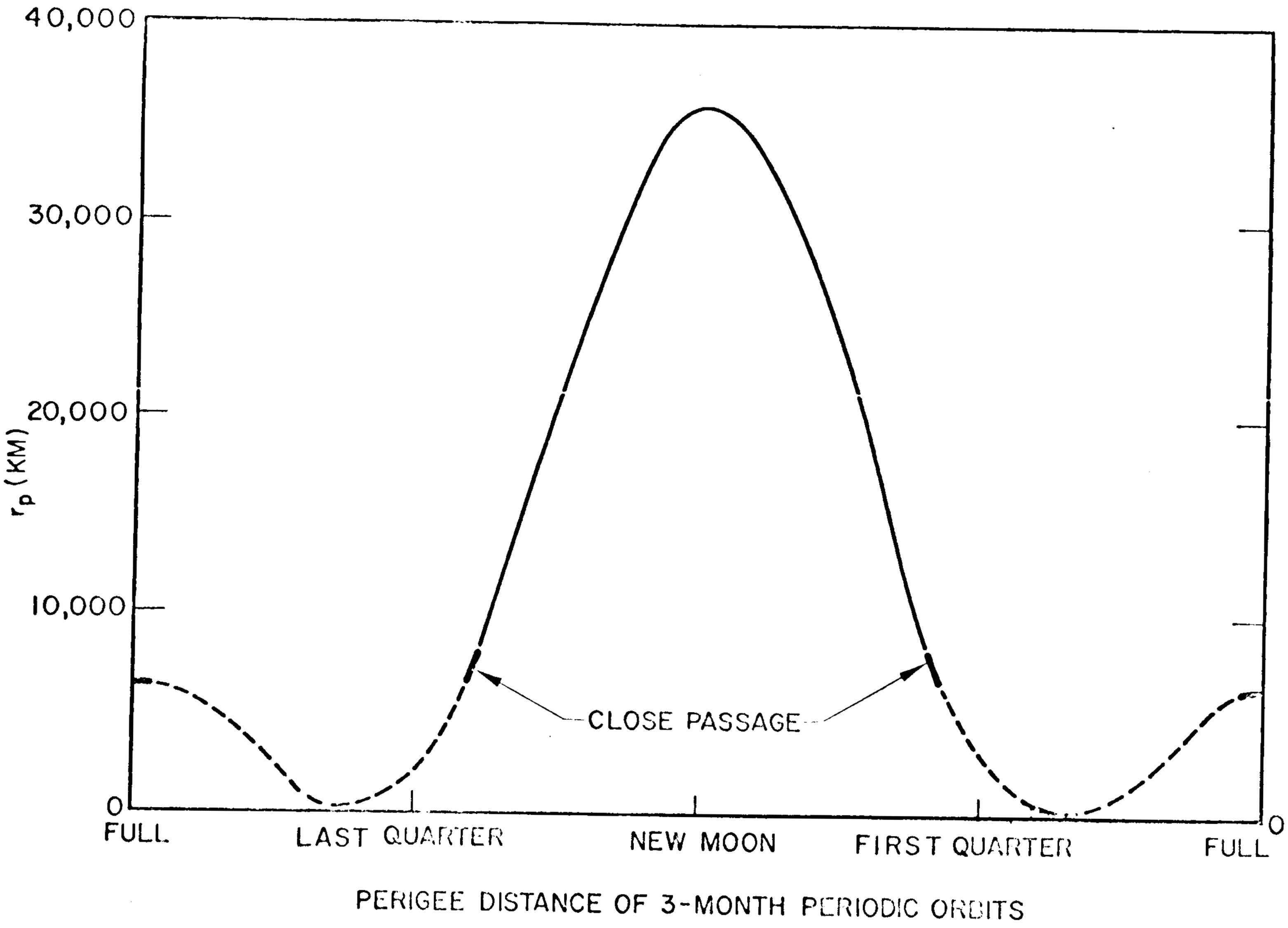


Fig. 6 Perigee Distance of 3-Month Periodic Orbits